THT.

THT WORKSHEETS

Learn With the Solutions

IB Diploma Programme

Mathematics: Analysis & Approaches

Higher Level

These solutions emphasise conceptual understanding, mathematical communication and interpretation. Students should be able to explain why each method works—not merely reproduce the calculation.

1.

Understanding the Question

We need to solve the quadratic equation:

$2x^2 - 7x + 3 = 0$

The expression can be factorised because we can find two binomials whose product produces the original quadratic.

Step 1: Factorise the quadratic

We need two numbers whose product is:

$2 \times 3 = 6$

and whose sum is:

-7

The numbers are -6 and -1.

Split the middle term:

$2x^2 - 6x - x + 3 = 0$

Group the terms:

$2x(x - 3) - 1(x - 3) = 0$

Factor out the common bracket:

$(2x - 1)(x - 3) = 0$

Step 2: Apply the zero-product property

If two factors multiply to zero, at least one factor must equal zero.

$2x - 1 = 0$

$2x = 1$

$x = 1/2$

or:

$x - 3 = 0$

$x = 3$

Check

For $x = 1/2$:

$2(1/2)^2 - 7(1/2) + 3$

$= 2(1/4) - 7/2 + 3$

$= 1/2 - 7/2 + 3$

$= -3 + 3$

$= 0$

For $x = 3$:

$2(3)^2 - 7(3) + 3$

$= 18 - 21 + 3$

$= 0$

Final Answer: $x = 1/2$ or $x = 3$

IB Tip

After factorising, verify that multiplying the brackets reproduces the original quadratic. This catches sign errors before solving.

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2.

Understanding the Question

The function is:

$f(x) = x^2 - 4x + 1$

We need to:

a) evaluate the function at $x = 5$;
b) determine the turning point.

a) Find f(5)

Substitute $x = 5$:

$f(5) = 5^2 - 4(5) + 1$

$= 25 - 20 + 1$

$= 6$

Final Answer: $f(5) = 6$

b) Find the turning point

Complete the square.

Start with:

$x^2 - 4x + 1$

Take half of -4:

$-4 \div 2 = -2$

Square it:

$(-2)^2 = 4$

Add and subtract 4:

$x^2 - 4x + 4 - 4 + 1$

The first three terms form a perfect square:

$(x - 2)^2 - 3$

Therefore:

$f(x) = (x - 2)^2 - 3$

This is vertex form:

$f(x) = (x - h)^2 + k$

The turning point is:

$(h, k) = (2, -3)$

Because the coefficient of the squared term is positive, the graph opens upwards, so this is a minimum point.

Final Answer: Turning point = (2, −3)

Key Idea

Completing the square reveals the horizontal and vertical translations of the basic graph $y = x^2$.

3.

Understanding the Question

The function is:

$g(x) = 3x - 5$

An inverse function reverses the original function.

We must:

a) find $g^{-1}(x)$;
b) verify the result through composition.

a) Find the inverse

Write:

$y = 3x - 5$

Swap x and y:

$x = 3y - 5$

Now solve for y.

Add 5:

$x + 5 = 3y$

Divide by 3:

$y = (x + 5)/3$

Therefore:

$\mathbf{g^{-1}(x) = (x + 5)/3}$

b) Verify using composition

Substitute $g^{-1}(x)$ into g:

$g(g^{-1}(x))$

$= 3[(x + 5)/3] - 5$

The factors of 3 cancel:

$= x + 5 - 5$

$= x$

Therefore:

$g(g^{-1}(x)) = x$

The inverse is verified.

Final Answer

$\mathbf{g^{-1}(x) = (x + 5)/3}$

and:

$\mathbf{g(g^{-1}(x)) = x}$

Common Mistake

Do not find the inverse by changing the sign of each term. The correct process is to interchange x and y and then solve for y.

4.

Understanding the Question

We need to differentiate:

$f(x) = 4x^3 - 9x^2 + 7$

Differentiation gives the instantaneous rate of change of the function.

Power Rule

For a term ax^n:

$d/dx(ax^n) = anx^{n-1}$

Differentiate each term

For $4x^3$:

$d/dx(4x^3) = 4 \times 3x^2 = 12x^2$

For $-9x^2$:

$d/dx(-9x^2) = -9 \times 2x = -18x$

For the constant 7:

$d/dx(7) = 0$

Therefore:

$f'(x) = 12x^2 - 18x$

Final Answer: $f'(x) = 12x^2 - 18x$

Common Mistake

The derivative of a constant is zero. Do not carry the +7 into the derivative.

5.

Understanding the Question

The position of a particle is:

$s(t) = 2t^3 - 9t^2 + 12t$

Velocity is the rate of change of position with respect to time.

Therefore:

$v(t) = ds/dt$

a) Find the velocity function

Differentiate each term:

$d/dt(2t^3) = 6t^2$

$d/dt(-9t^2) = -18t$

$d/dt(12t) = 12$

Therefore:

$v(t) = 6t^2 - 18t + 12$

Final Answer: $v(t) = 6t^2 - 18t + 12$

b) Find the velocity when $t = 2$

Substitute $t = 2$:

$v(2) = 6(2)^2 - 18(2) + 12$

$= 6(4) - 36 + 12$

$= 24 - 36 + 12$

$= 0$

Final Answer: $v(2) = 0$ m/s

Interpretation

At $t = 2$ seconds, the particle is momentarily stationary.

This does not necessarily mean that it remains at rest. Its direction may change at that instant.

IB Tip

Always interpret a zero velocity in context. It indicates an instantaneous stationary point, not necessarily that the particle has stopped permanently.

6.

Understanding the Question

The basic function is:

$f(x) = x^2$

It is transformed into:

$g(x) = (x - 2)^2 + 5$

This is already written in vertex form:

$g(x) = (x - h)^2 + k$

where the turning point is (h, k).

a) Turning point

Compare:

$g(x) = (x - 2)^2 + 5$

with:

$g(x) = (x - h)^2 + k$

Therefore:

$h = 2$

$k = 5$

Final Answer: Turning point = (2, 5)

b) Axis of symmetry

The vertical line through the turning point is:

$x = 2$

Final Answer: Axis of symmetry = x = 2

c) Describe the transformation

Starting from:

$y = x^2$

The term:

$(x - 2)$

moves the graph 2 units to the right.

The +5 outside the square moves the graph 5 units upwards.

The shape and orientation of the parabola do not change because the coefficient of the squared term remains 1.

Final Answer

The graph is translated:

- **2 units to the right**
- **5 units upwards**

Common Mistake

The sign inside the bracket acts in the opposite direction. The expression $(x - 2)$ shifts the graph right, not left.

7.

Understanding the Question

We need to expand:

$(2x - 1)^3$

This means:

$(2x - 1)(2x - 1)(2x - 1)$

We can use the binomial identity:

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Let:

$a = 2x$

$b = 1$

Apply the identity

$(2x - 1)^3$

$= (2x)^3 - 3(2x)^2(1) + 3(2x)(1)^2 - 1^3$

Calculate each term:

$(2x)^3 = 8x^3$

$-3(2x)^2 = -3(4x^2) = -12x^2$

$3(2x) = 6x$

$-1^3 = -1$

Therefore:

Final Answer: $8x^3 - 12x^2 + 6x - 1$

Check

The coefficients 1, 3, 3, 1 come from the third row of Pascal's triangle, with alternating signs because the binomial contains subtraction.

8.

Understanding the Question

We need to solve:

$3x - 7 \leq 2x + 5$

An inequality is solved similarly to an equation, but the final answer represents a range of values.

Step 1: Collect the x-terms

Subtract 2x from both sides:

$3x - 2x - 7 \leq 5$

$x - 7 \leq 5$

Step 2: Isolate x

Add 7 to both sides:

$x \leq 12$

Final Answer: $x \leq 12$

Number-line representation

- Place a **closed circle** at 12 because 12 is included.
- Shade the line to the **left** because the solution includes values less than 12.

Common Mistake

The inequality sign reverses only when multiplying or dividing by a negative number. No reversal is required here.

9.

Understanding the Question

The population model is:

$P(t) = 18{,}000(1.025)^t$

where t is measured in years.

The initial population is 18,000, and the factor 1.025 represents annual growth of 2.5%.

a) Population after 8 years

Substitute $t = 8$:

$P(8) = 18{,}000(1.025)^8$

Calculate the growth factor:

$(1.025)^8 \approx 1.2184029$

Therefore:

$P(8) \approx 18{,}000 \times 1.2184029$

$P(8) \approx 21{,}931.25$

A population must be expressed as a whole number.

Final Answer: Approximately 21,931 people

b) Limitation of the model

The model assumes that the population grows by exactly 2.5% every year.

In reality, population growth may change because of:

- migration;
- birth rates;
- death rates;
- employment opportunities;

- housing availability;
- government policy;
- economic or environmental events.

Sample Answer

One limitation is that the model assumes a constant percentage growth rate. Actual population growth is unlikely to remain exactly 2.5% every year.

Final Answer: The constant annual growth assumption may not reflect real population changes.

Key Idea

A mathematical model simplifies reality. Its predictions are useful only while its assumptions remain reasonable.

10.

Understanding the Question

The graph is:

$y = x^2 - 6x + 5$

The x-intercepts occur where:

$y = 0$

Therefore, solve:

$x^2 - 6x + 5 = 0$

Factorise

We need two numbers that:

- multiply to +5;
- add to −6.

The numbers are −1 and −5.

Therefore:

$x^2 - 6x + 5$

$= (x - 1)(x - 5)$

Solve

$(x - 1)(x - 5) = 0$

So:

$x - 1 = 0$

$x = 1$

or:

$x - 5 = 0$

$x = 5$

At an x-intercept, $y = 0$.

Final Answer: (1, 0) and (5, 0)

Common Mistake

The solutions $x = 1$ and $x = 5$ are x-values. The question asks for coordinates, so include $y = 0$.

11.

Understanding the Question

The sequence is:

5, 11, 17, 23, …

The difference between consecutive terms is constant, so this is an arithmetic sequence.

a) Common difference

$11 - 5 = 6$

$17 - 11 = 6$

$23 - 17 = 6$

Final Answer: Common difference = 6

b) Formula for the nth term

For an arithmetic sequence:

$a_n = a_1 + (n - 1)d$

Here:

$a_1 = 5$

$d = 6$

Substitute:

$a_n = 5 + (n - 1)6$

Expand:

$a_n = 5 + 6n - 6$

$a_n = 6n - 1$

Final Answer: $a_n = 6n - 1$

c) Find the 25th term

Substitute $n = 25$:

$a_{25} = 6(25) - 1$

$= 150 - 1$

$= 149$

Final Answer: 149

Check

For $n = 1$:

$a_1 = 6(1) - 1 = 5$

The formula reproduces the first term.

12.

Understanding the Question

The probability of a faulty component is:

0.04

The laboratory produces:

120 components

The expected number is found by multiplying the number of trials by the probability of the event.

Calculate the expected number

Expected number:

120×0.04

$= 4.8$

Final Answer: Expected number = 4.8 faulty components

In practical terms, this suggests approximately 5 faulty components.

Why might the actual number differ?

An expected value is a long-run average. It does not predict the exact result of a single production batch.

For example, one batch may contain:

- 3 faulty components;
- 5 faulty components;
- 7 faulty components.

Over many large batches, the average number should approach 4.8.

Final Answer: The actual number can differ because probability describes expected long-term behaviour, not an exact outcome for one sample.

Precision Note

The expected value may be non-integer even though the actual number of faulty components must be a whole number.

13.

Understanding the Question

The monthly profit is modelled by:

$P(x) = -2x^2 + 160x - 1800$

This is a quadratic with a negative coefficient of x^2, so its graph opens downwards.

Therefore, the turning point gives the maximum profit.

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The worksheet defines x as the number of products sold **in hundreds**.

a) Find the value of x giving maximum profit

For:

$P(x) = ax^2 + bx + c$

the x-coordinate of the vertex is:

$x = -b/(2a)$

Here:

$a = -2$

$b = 160$

Substitute:

$x = -160/[2(-2)]$

$= -160/-4$

$= 40$

Final Answer: x = 40

Because x is measured in hundreds, this corresponds to:

$40 \times 100 = 4{,}000$ products

b) Determine the maximum profit

Substitute $x = 40$:

$P(40) = -2(40)^2 + 160(40) - 1800$

Calculate the square:

$40^2 = 1600$

Then:

$P(40) = -2(1600) + 6400 - 1800$

$= -3200 + 6400 - 1800$

$= 3200 - 1800$

$= 1400$

Final Answer: Maximum profit = $1,400

Interpretation

According to the model, the company achieves maximum monthly profit when it sells 4,000 products.

Common Mistake

Do not report 40 products. The question defines x in hundreds of products.

14.

Understanding the Question

The line is:

$2x - y + 7 = 0$

We need:

a) its gradient;
b) the equation of a perpendicular line through (3, 4).

a) Find the gradient

Rearrange into the form:

$y = mx + c$

Start with:

$2x - y + 7 = 0$

Subtract 2x and 7 from both sides:

$-y = -2x - 7$

Multiply by −1:

$y = 2x + 7$

Therefore, the gradient is:

$m = 2$

Final Answer: Gradient = 2

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b) Find the perpendicular line

Perpendicular gradients satisfy:

$m_1 m_2 = -1$

The original gradient is 2.

Therefore, the perpendicular gradient is:

$m_2 = -1/2$

Use point-slope form:

$y - y_1 = m(x - x_1)$

Substitute the point (3, 4):

$y - 4 = -1/2(x - 3)$

This is already a valid equation.

To write it in slope-intercept form, expand:

$y - 4 = -1/2x + 3/2$

Add 4:

$y = -1/2x + 3/2 + 4$

Write 4 as 8/2:

$y = -1/2x + 11/2$

Final Answer: $y = -1/2x + 11/2$

Check

Substitute (3, 4):

Right-hand side:

$-1/2(3) + 11/2$

$= -3/2 + 11/2$

$= 8/2$

$= 4$

The line passes through the required point.

15.

Understanding the Question

The claim is:

"Every differentiable function is continuous."

This is a theoretical statement about the relationship between differentiability and continuity.

Explanation

If a function is differentiable at a point, the derivative exists at that point.

For the derivative to exist, the function must approach a single finite value from both sides and agree with the function value.

Therefore, differentiability implies continuity.

Symbolically:

Differentiable $\Rightarrow$ Continuous

However, the converse is not always true.

A function may be continuous but not differentiable.

Counterexample to the converse

Consider:

$f(x) = |x|$

This function is continuous at $x = 0$.

However, it has a sharp corner at $x = 0$:

- the left-hand derivative is -1;
- the right-hand derivative is 1.

Because the one-sided derivatives differ, the function is not differentiable there.

Final Answer: The statement is true. Every differentiable function is continuous at the point of differentiability.

IB Precision

The correct implication is:

Differentiability implies continuity.

Continuity does **not** necessarily imply differentiability.

Quick Answer Check

1. $x = 1/2$ or 3
2. a) 6
   b) Turning point = $(2, -3)$
3. a) $g^{-1}(x) = (x + 5)/3$
   b) $g(g^{-1}(x)) = x$
4. $f'(x) = 12x^2 - 18x$
5. a) $v(t) = 6t^2 - 18t + 12$
   b) $v(2) = 0$ m/s
6. a) $(2, 5)$
   b) $x = 2$
   c) Translated 2 units right and 5 units up
7. $8x^3 - 12x^2 + 6x - 1$
8. $x \leq 12$
9. a) Approximately 21,931 people
   b) The model assumes a constant annual growth rate
10. $(1, 0)$ and $(5, 0)$
11. a) 6
    b) $a_n = 6n - 1$
    c) 149
12. Expected value = 4.8, or approximately 5 faulty components; the actual result may differ through random variation
13. a) $x = 40$, representing 4,000 products
    b) Maximum profit = $1,400
14. a) Gradient = 2
    b) $y = -1/2x + 11/2$
15. True; differentiability implies continuity.

